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THE FOUNDATIONS OF GEOMETRY.

[CONCLUDED.]

DIFFERENT GEOMETRICAL SYSTEMS.

STRAIGHTNESS, flatness, and rectangularity are qualities which cannot (like numbers) be determined in purely quantitative terms. A right angle is not an arbitrary amount of ninety degrees, but the quarter of a circle, and the nature of angles and degrees is not derivable either from arithmetic or from pure reason. A plane is not zero, but a zero of curvature in a boundary between two solids. There is a qualitative element which cannot be expressed in numbers alone; and this qualitative element is determined, as Kant would express it by *Anschauung*, or as we prefer to say by pure motility, viz., it belongs to the domain of the *a priori* of doing.

Our method of creating by construction the straight line, the plane, and the right angle, does not exclude the possibility of other methods of space-measurement, the standards of which would not be even boundaries, such as straight lines, but lines possessed of either a positive curvature like the sphere or a negative curvature rendering their surface pseudo-spherical.

Spheres are well known and do not stand in need of description. Their curvature which is positive is determined by the reciprocal of their radius.

Pseudo-spheres are surfaces of negative curvature, and pseudo-spherical surfaces are saddle-shaped. Only limited pieces can be connectedly represented, and we reproduce from Helmholtz, *loc. cit.*, p. 42, two instances. If arc *ab* in figure 1 revolves round an axis *AB*, it will describe a concave-convex surface like that of the inside of a wedding-ring; and in the same way, if either of the curves of figure 2 revolve round their axis of symmetry, it will describe one half of a pseudospherical surface resembling the shape of a morning-glory whose tapering stem is infinitely prolonged. Helmholtz compares the former to an anchor-ring, the latter to a champagne glass.

The sum of the angles of triangles on spheres always exceeds 180°, and the larger the sphere the more will their triangles resemble the triangle in the plane. On the other hand, the sum of the angles of triangles on the pseudosphere will always be somewhat less than 180°. If we define the right angle as the fourth part of a whole circuit, it will be seen that analogously the right angle in the plane differs from the right angles on the sphere as well as the pseudosphere.

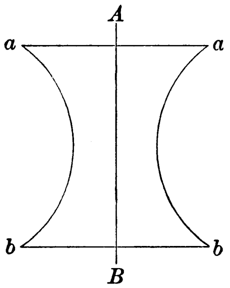


Fig. 1.

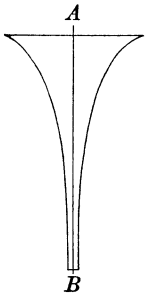


Fig. 2.

We may add that while in spherical space several shortest lines are possible, in pseudospherical space we can draw one shortest line only. Both surfaces, however, are homogeneous (i. e., figures can be moved in it without suffering a change in dimensions), but the parallel lines which do not meet are impossible in either.

We may further construct surfaces in which changes of place involve either expansion or contraction, but it is obvious that they would be less serviceable as systems of space-measurement the more irregular they grow.

DIMENSIONS AS CO-ORDINATES.

But why is space tri-dimensional? Why is it not four or five or n-dimensional?

Let us ask first what "dimension" means.

Does dimension mean direction? Obviously not, for we have seen that the possibilities of direction in space are infinite.

Dimension is only a popular term for coördinate. In space there are no dimensions laid down, but in a space of infinite directions three coördinates are needed to determine from a given point of reference the position of any other point.

We have halved space and produced a plane, P_1; we have halved the plane P_1 and produced the straight line AB on the plane. We have halved the straight line, the even boundary between the two half planes, by the perpendicular CD, in the point O, thus producing right angles on the straight line AB, represented in the cross-creases of the twice folded sheet of paper.

The point is simply a place, or as the mathematical term is, a *locus*, viz., a spot in space the site of which is determined by the crossing of two lines. It has no extension, and here our method of producing even boundaries by halving comes to a natural end. So far our products are the plane, the straight line, the point, and as an incidental but valuable by-product, the right angle.

We may now venture on a synthesis of our materials. We lay two planes, P_2 and P_3, through the two creases at right angles on the original plane P_1, represented by the sheet of paper, and it becomes apparent that the two new planes P_2 and P_3 will intersect at O, producing a line EF common to both planes P_2 and P_3, and they will bear the same relation to each as each one does to the original plane P_1, that is to say: the whole system is congruent with itself. If we make the planes change places, P_1 may as well take the place of P_2 and P_2 of P_3 and P_3 of P_2 or P_1 of P_3, etc., or *vice versa*, and all the internal relations would remain absolutely the same. Accordingly we have here in this system of the three planes at right angles (the result of repeated halving), a composition of even boundaries which, as the simplest and least complicated construction of its kind, recommends itself for a standard of measurement of the whole spread of motility.

TRIDIMENSIONALITY.

The most significant feature of our construction consists in this, that we thereby produce a convenient system of reference for determining *all* points in coördinates of straight lines standing at right angles to the three planes.

If we start from the ready conception of objective space (the juxtaposition of things or of loci) we can produce the same result of three planes at right angles by a process of halving. We cut space in two equal halves by the horizontal plane P_1. We repeat the cutting so as to let the two halves of the first cut in their angular relation to the new cut (in P_2) be congruent with each other, a procedure which is possible only if we make use of the even boundary concept with which we have become acquainted. Accordingly, the second cut should stand at right angles on the first cut. The two planes P_1 and P_2 have one line in common, EF, and any plane placed at right angles to EF (in the point O) will again satisfy the demand of dividing space, including the two planes P_1 and P_2, into two congruent halves. The two new lines, produced by the cut of the third plane P_3 through the two former planes P_1 and P_2, stand both at right angles to EF. Should we continue our method of cutting space at right angles in O on either of these lines, we would produce a plane coincident with P_1, which is to say, that the possibilities of the system are exhausted.

This implies that in any system of pure space *three coördinates are sufficient* for the determination of any place from a given reference point.

The number three is a concept of boundary as much as the straight line. Under specially complicated conditions we might need ten coördinates to calculate the place of a point, but in empty space the number three, the lowest number that is really and truly a number, is sufficient. If space is to be empty space constructed as anyness for the purpose of determining juxtaposition, three co-ordinates are sufficient, because our system of reference consists of three planes, and we have seen above that there is no possibility of introducing a fourth plane without destroying its character of being congruent with itself, which imparts to it the simplicity that renders it available for a standard of measurement.

Three is a peculiar number which is of great significance. It is the first real number, being the simplest multiplex. One and two are not yet numbers in the full sense of the word. One is the unit, two is a couple or a pair, but three is the smallest amount of a genuine plurality. Savages who can distinguish only between one and two have not yet evolved the notion of number; and the transition to the next higher stage involving the knowledge of "three" passes through a mental condition in which there exists only the notion one, two, and plurality of any kind. When the idea of three is once definitely recognised, the naming of all other numbers can follow in rapid succession.

Certainly it is no accident that in order to construct the simplest figure which is a real figure, at least three lines are needed. The importance of the triangle, which becomes most prominent in trigonometry, is due to its being the simplest possible figure which accordingly possesses the intrinsic worth of economy.

The number three plays also a significant part in logic, and in the branches of the applied sciences, and thus we need not be astonished at finding the three held in religious reverence, for the doctrine of the Trinity has its basis in the constitution of the universe and can be fully justified by the laws of pure form.

THE APPARENT ARBITRARINESS OF THE A PRIORI.

Since Riemann has generalised the conception of space, the tridimensionality of space seems very arbitrary.

Why are three coördinates sufficient for pure space determinations? The obvious answer is, Because we have three planes in our construction of space-boundaries. We might as well ask, why do the three planes cut the entire space into 8 equal parts; the simple answer is that we have halved space three times, and $2^3 = 8$.

The reason is practically the same as the simpler one, why have we two halves if we divide an apple into two equal parts.

These answers are simple enough, but there is another question which here seems in order: Why not continue the method of halving? And there is no other answer than that it is impossible. The two superadded planes P_2 and P_3 both standing at right angles to the original plane, necessarily halve each other, and thus the four right angles of each plane P_2 and P_3 on the centre of intersection, form a complete plane for the same reason that four quarters are one whole. We have in each case four quarters, and $\frac{4}{4}=1$.

Purely logical arguments (i. e., all modes of reasoning that are rigorously *a priori*, the *a priori* of abstract being) break down and we must resort to the methods of the *a priori* of doing. We cannot understand or grant the argument without admitting the conception of space, previously created by a spread of pure motion. Kant would say that we need here the data of *reine Anschauung*, and Kant's *reine Anschauung* is a product of our motility. As soon as we admit that there is an *a priori* of doing (of free motility) and that our conception of pure space and time (Kant's *reine Anschauung*) is its product, we understand that our mathematical conceptions cannot be derived from pure logic alone but must finally depend upon our motility, viz., the function that begets our notion of space.

If we divide an apple by a vertical cut through its center, we have two halves. If we cut it again by a horizontal cut through its center, we have four quarters. If we cut it again with a cut that is at right angles to both prior cuts, we have eight eighths. It is obviously impossible to insert among these three cuts a fourth cut that would stand at right angles to these others. The fourth cut through the center, if we needs would have to make it, will fall into one of the prior cuts and be a mere repetition of it, producing no result; or if we made it slanting, it would not cut all eight parts but only four of them; it would not produce sixteen equal parts, but twelve unequal parts, viz., eight sixteenths plus four quarters.

If we do not resort to a contemplation of the scope of motion, if we neglect to represent in our imagination the figure of the three planes and rely on pure reason alone (i. e., the rigid *a priori*), we have no means of refuting the assumption that we ought to be able to continue halving the planes by other planes at right angles. Yet is the proposition as inconsistent as to expect that there should be regular pentagons, or hexagons, or triangles, the angles of which are all ninety degrees.

From the standpoint of pure reason alone we cannot disprove the incompatibility of the idea of a rectangular pentagon. If we insist on constructing by hook or crook a rectangular pentagon, we will succeed, but we must break away from the straight line or the plane. A rectangular pentagon is not absolutely impossible; it is absolutely impossible in the plane; and if we produce one, it will be twisted.

Such was the result of Lobatchevski's and Bolyai's construction of a system of geometry in which the theorem of parallels does not hold. Their geometries cease to be even; they are no longer Euclidean and render the even boundary conceptions unavailable as standards of measurement. That is all.

If by logic we understand consistency, and if anything that is self-contradictory and incompatible with its own nature be called illogical, we would say that it is not the logic of pure reason that renders certain things impossible in our geometric constructions, but the logic of our scope of motion. The latter introduces a factor which determines the nature of geometry, and if this factor is neglected or misunderstood, the fundamental notions of geometry must appear arbitrary.

DEFINITENESS OF CONSTRUCTION.

The problems which puzzle some of the metaphysicians of geometry seem to have one common foundation, which is the definiteness of geometrical construction. Geometry starts from empty nothingness, and we are confronted with rigid conditions which it does not lie in our power to change. We make a construction, and the result is something new, perhaps something which we have not intended, something at which we are surprised. The synthesis is a child of our own doing, yet there is an objective element in it over which we have no command, and this objective element is rigid, uncompromising, an irrefragable necessity, a stubborn fact, immovable, inflexible, immutable. What is it?

Our metageometricians overlook the fact that their nothingness is not an absolute nothing, but only an absence of concreteness. If they make definite constructions, they must (if they only remain consistent) expect definite results. This definiteness is the logic that dominates their operations. Sometimes the results seem arbitrary, but they never are; for they are necessary, and all questions *why?* can elicit only answers that turn in a circle and are mere tautologies.

Why, we may ask, do two straight lines, if they intersect, produce four angles? Perhaps we did not mean to construct angles, but here we have them in spite of ourselves.

And why is the sum of these four angles equal to 360°? Why, if two are acute, will the other two be found to be obtuse? Why, if one angle be a right angle, will all four be right angles? Why will the sum of any two adjacent angles be equal to two right angles? etc. Perhaps we should have preferred three angles only, or four acute angles, but we cannot have them, at least not by this construction.

We have seen that the tridimensionality of space is arbitrary only if we judge of it as a notion of pure reason, without taking into consideration the method of its construction as a scope of mobility. Tridimensionality is only one instance of apparent arbitrariness among many others of the same kind.

We cannot construct a figure of two straight lines, and we cannot construct a solid of three even surfaces.

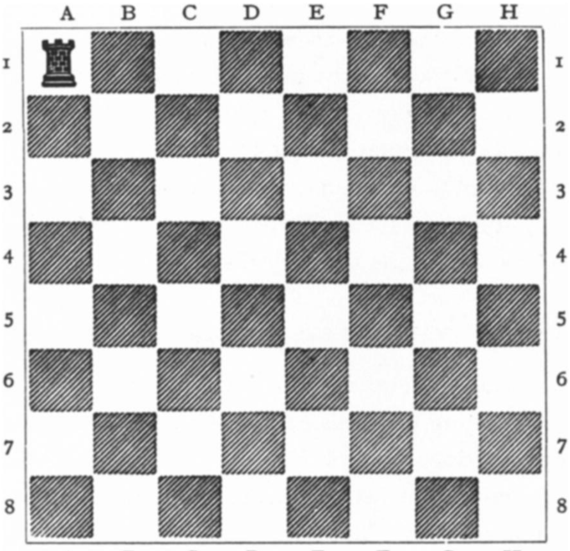
There are only definite forms of polyhedra possible, and the surfaces of every one are definitely determined. To the mind uninitiated into the secrets of mathematics it would seem arbitrary that there are two hexahedra (viz., the cube and the duplicated tetrahedron), while there is no heptahedron. And why can we not have an octahedron with quadrilateral surfaces? We might as well ask, why is the square not round!

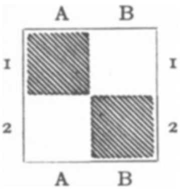
Prof. G. B. Halsted says in the Translator's Appendix to his English edition of Lobatchevsky's *Theory of Parallels*, p. 48:

> "But is it not absurd to speak of space as interfering with anything? If you think so, take a knife and a raw potato and try to cut it into a seven-edged solid."

Truly Professor Halsted's contention, that the laws of space interfere with our operations, is true. Yet it is not space that squeezes us, but the laws of construction determine the shape of the figures which we make.

A simple instance that illustrates the way in which space interferes with our plans and movements is the impossible demand on the chess-board to start a rook in one corner (*A* 1) and pass it with the rook motion over all fields once, but only once, and let it end its journey on the opposite corner (*H* 8). Rightly considered it is not space that interferes with our mode of action, but the law of consistency. The proposition does not contain anything illogical; the words are quite rational and the sentences grammatically correct. Yet is the task impossible, because we cannot turn to the





right and left at once, nor can we be in two places at once, neither can we undo an act once done or for the nonce change the rook into a bishop; but something of that kind would have to be done, if we start from *A* 1 and passing with the rook motion through *A* 2 and *B* 1 over to *B* 2. In other words: Though the demand is not in conflict with the logic of abstract being or the grammar of thinking, it is impossible because it collides with the logic of doing; the logic of moving about, the *a priori* of motility.

The same is true of the squaring of the circle.

We cannot venture on self-contradictory enterprises without being defeated, and if the relation of the circumference to the diameter is an infinite transcendent *series*, being

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \tfrac{1}{15} + \tfrac{1}{17} - \tfrac{1}{19} + \tfrac{1}{21} - \tfrac{1}{23} + \cdots$$

we cannot expect to square the circle.

If we compute the series, π becomes 3.14159265...., figures which seem as arbitrary as the most whimsical fancy.

It does not seem less strange that $e = 2.71828$; and yet it is as little arbitrary as the equation $3 \times 4 = 12$.

The definiteness of our mathematical constructions and arithmetical computations is based upon the inexorable law of determinism, and everything is fixed by the mode of its construction.

ONE SPACE, BUT VARIOUS SYSTEMS OF SPACE-MEASUREMENT.

Riemann has generalised the idea of space and would thus justify us in speaking of "spaces." The common notion of space, which is the traditional one of Euclidean geometers, has been degraded into a mere species of space, one possible instance among many other possibilities. And its very legitimacy has been doubted, for it has come to be looked upon in some quarters as only a popular (not to say vulgar and commonplace) notion, a mere working hypothesis, infested with many arbitrary conditions of which the ideal conception of absolute space should be free. How much more interesting and aristocratic are curved space, the dainty two-dimensional space, and above all the four-dimensional space with its magic powers!

The new space-conception seems bewildering. Some of these new spaces are constructions that are not concretely representable, but only abstractly thinkable; yet they allow us to indulge in ingenious dreams. Think only of two-dimensional beings, and how limited they are! They can have no conception of a third dimension! Then think of four-dimensional beings; how superior they must be to us poor tridimensional beings! As we can take a figure situated within a circle through the third dimension and put it down again outside the circle without crossing the circumference, so four-dimensional beings could take tridimensional things encased in a box from their hiding-place and put them back on some other spot. They could easily help themselves to all the money in the steel-lined safes of our banks, and they could perform the most difficult obstetrical feats without resorting to the dangerous Cæsarean operation.

Curved space is not less interesting. Just as light may pass through a medium that offers such a resistance as will involve a continuous displacement of the rays, so in curved space the lines of greatest intensity would be subject to a continuous modification. The beings of curved space may be assumed to have no conception of truly straight lines. They must deem it quite natural that if they walk on in the straightest possible manner they will finally but unfailingly come back to the same spot. Their world-space is not as vague and mystical as ours: it is not infinite, hazy at a distance, vague and without end, but definite, well rounded off, and perfect. Presumably their lives have the same advantages moving in boundless circles, while our progression in straight lines hangs between two infinitudes—the past and the future!

All these considerations are very interesting and open new vistas to imaginative speculators and inventors, and we cannot deny that the generalisation of our space-conception has proved helpful by throwing new light upon geometrical problems and widening the horizon of our mathematical knowledge.

Nevertheless after a mature deliberation of Riemann's proposition, I have come to the conclusion that it leads us off in a wrong direction, and in contrast to his conception of space as being one instance among many possibilities, I would insist upon the uniqueness of space. Space is the possibility of motion in all directions, and mathematical space is the ideal construction of our scope of motion in all directions.

The homogeneity of space is due to our abstraction which omits all particularities, and its homaloidality means only that straight lines are possible and will best serve us as standards of measurement.

Curved space, so called, is a more complicated construction of space-measurement to which some additional feature of a particular nature has been admitted, and in which we waive the advantages of even boundaries as means of measurement.

Space, the actual scope of motion, remains different from all systems of space-measurement, be they homaloidal or curved, and should not be subsumed with them under one and the same category.

Riemann's several space-conceptions are not spaces in the proper sense of the word, but systems of space-measurement. It is true that space is a tridimensional manifold, and a plane a two-dimensional manifold, and we can think of other systems of n manifoldness; but for that reason all these different manifolds do not become spaces. Man is a mammal having two prehensiles (his hands); the elephant is a mammal with one prehensile (his trunk); tailless monkeys like the pavian have four; and tailed monkeys have five prehensiles. Is there any logic in extending the denomination man to all these animals, and should we define the elephant as a man with one prehensile, the pavian as a man with four prehensiles and tailed monkeys as men with five prehensiles? Our zoölogists would at once protest and denounce it as an illogical misuse of names.

Space is a manifold, but not every manifold is a space.

Of course every one has a right to define the terms he uses, and obviously my protest simply rejects Riemann's use of a word, but I claim that his identification of "space" with "manifold" is the source of inextricable confusion.

It is well known that all colors can be reduced to three primary colors, yellow, red, and blue, and thus we can determine any possible tint by three coördinates, and color just as much as mathematical space is a threefold, viz., a system in which three coördinates are needed for the determination of any thing. But because color is a threefold, no one would assume that color is space.

Riemann's manifolds are systems of measurement, and the system of three coördinates on three intersecting planes is an *a priori* or pure formal and ideal construction invented to calculate space. We can invent other more complicated systems of measurement, with curved lines and with more than three or less than three coördinates. We can even employ them for space-measurement, although they are rather awkward and unserviceable; but these systems of measurement are not "spaces" and if they are called so, they are spaces by courtesy only. By a metaphorical extension we allow the idea of system of space-measurement to stand for space itself. It is a brilliant idea and quite as ingenious as the invention of animal fables in which our quadruped fellow-beings are endowed with speech and treated as human beings. But such poetical licences, in which facts are stretched and the meaning of terms is slightly modified, is possible only if instead of the old-fashioned straight rules of logic we grant a slight curvature to our syllogisms.

FICTITIOUS SPACES AND THE APRIORITY OF ALL SYSTEMS OF SPACE-MEASUREMENT.

Mathematical space, so called, is strictly speaking no space at all, but the mental construction of a manifold, being a tridimensional system of space-measurement invented for the determination of actual space.

Neither can a manifold of two dimensions be called a space. It is a mere boundary in space, it is no reality, but a concept, a construction of pure thought.

Further, the manifold of four dimensions is a system of measurement applicable to any reality for the determination of which four coördinates are needed. It is applicable to real space if there is connected with it in addition to the three planes at right angles another condition of a constant nature, such as gravity.

At any rate, we must deny the applicability of a system of four dimensions to empty space void of any such particularity. Unless Riemann can show that sixteen cubes (neither more nor fewer) could be so placed together that they have one point of common contact, I must regard the idea of four-dimensional space as chimerical, the word space being used in the common acceptance of the term as juxtaposition or as the scope of motion. So long as four quarters make one whole, and four right angles make one entire circumference, and so long as the contents of a sphere which covers the entire scope of motion round its center equals $\frac{4}{3}\pi r^3$, there is no sense in entertaining the idea that empty space might be four-dimensional.

But the argument is made and sustained by Helmholtz that as two dimensional beings perceive two dimensions only and are unable to think how a third dimension is at all possible, so we tridimensional beings cannot represent in thought the possibility of a fourth dimension. Helmholtz, speaking of beings of only two dimensions living on the surface of a solid body, says:

> "If such beings worked out a geometry, they would of course assign only two dimensions to their space. They would ascertain that a point in moving describes a line, and that a line in moving describes a surface. But they could as little represent to themselves what further spatial construction would be generated by a surface moving out of itself, as we can represent what should be generated by a solid moving out of the space we know. By the much-abused expression 'to represent' or 'to be able to think how something happens' I understand—and I do not see how anything else can be understood by it without loss of all meaning—the power of imagining the whole series of sensible impressions that would be had in such a case. Now, as no sensible impression is known relating to such an unheard-of event, as the movement to a fourth dimension would be to us, or as a movement to our third dimension would be to the inhabitants of a surface, such a 'representation' is as impossible as the 'representation' of colors would be to one born blind, if a description of them in general terms could be given to him.
>
> "Our surface-beings would also be able to draw shortest lines in their superficial space. These would not necessarily be straight lines in our sense, but what

> are technically called *geodetic lines* of the surface on which they live; lines such as are described by a *tense* thread laid along the surface, and which can slide upon it freely."....
>
> "Now, if beings of this kind lived on an infinite plane, their geometry would be exactly the same as our planimetry. They would affirm that only one straight line is possible between two points; that through a third point lying without this line only one line can be drawn parallel to it; that the ends of a straight line never meet though it is produced to infinity, and so on."....
>
> "But intelligent beings of the kind supposed might also live on the surface of a sphere. Their shortest or straightest line between two points would then be an arc of the great circle passing through them."....
>
> "Of parallel lines the sphere-dwellers would know nothing. They would maintain that any two straightest lines, sufficiently produced, must finally cut not in one only but in two points. The sum of the angles of a triangle would be always greater than two right angles, increasing as the surface of the triangle grew greater. They could thus have no conception of geometrical similarity between greater and smaller figures of the same kind, for with them a greater triangle must have different angles from a smaller one. Their space would be unlimited, but would be found to be finite or at least represented as such.
>
> "It is clear, then, that such beings must set up a very different system of geometrical axioms from that of the inhabitants of a plane, or from ours with our space of three dimensions, though the logical powers of all were the same."

I deny what Helmholtz implicitly assumes that sensible impressions are needed to enable us to represent purely formal relations. Thus we have the idea of a surface as a boundary between solids, but surfaces do not exist in reality. All real objects are solid, and our idea of surface is a mere abstraction. Two dimensional things are unreal, and yet we form the notion of surfaces, and lines, and points, and pure space, etc. There is no straight line in existence, yet we have the notion of a straight line. The straight lines on paper are incorrect pictures of the true straight lines which are purely ideal constructions. Our *a priori* constructions are not a product of our sense-impressions, but are independent of sense or anything sensed. They are purely formal, and even if we grant that two-dimensional beings were possible, we would have no reason to assume that two-dimensional beings could not construct a tridimensional space-conception.

Two-dimensional beings could not be possessed of a material body, because their absolute flatness substantially reduces their shape to nothingness. But if they existed, they would be limited to movements in two directions and thus must be expected to be incredulous as to the possibility of jumping out of their flat existence and returning into it through a third dimension. Having never moved in a third dimension, they could speak of it as the blind might discuss colors; in their flat minds they could have no true conception of its significance and would be unable clearly to picture it in their imagination; but for all their limitations, they could very well develop the abstract idea of tri-dimensional space and therefrom derive all particulars of its laws and conditions and possibilities.

We ourselves are tridimensional; we can measure the space in which we move with three coördinates, yet we can definitely say that if space were four-dimensional, a body constructed of two factors, so as to have a four-dimensional solidity, would be expressed in the formula:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

We can calculate, compute, excogitate, and describe all the characteristics of four-dimensional space, so long as we remain in the realm of abstract thought and do not venture to make use of our motility and execute our plan in an actualised construction of motion. From the standpoint of pure logic, there is nothing irrational about the assumption; but as soon as we make an *a priori* construction of the scope of our motility, we find out the incompatibility of the whole scheme. Helmholtz continues:

> "But let us proceed still farther.
>
> "Let us think of reasoning beings existing on the surface of an egg-shaped body. Shortest lines could be drawn between three points of such a surface and a triangle constructed. But if the attempt were made to construct congruent triangles at different parts of the surface, it would be found that two triangles, with three pairs of equal sides, would not have their angles equal."

If there were two-dimensional beings living on an egg-shell, they would most likely have to determine the place of their habitat by experience just as much as we tridimensional beings living on a flattened sphere have to map out our world by measurements made *a posteriori* and based upon *a priori* systems of measurement.

If the several systems of space-measurement were not *a priori* constructions, how could Helmholtz who does not belong to the class of two-dimensional beings tell us what their notions must be like?

I claim that if there were surface beings on a sphere or on an egg-shell, they would have the same *a priori* notions as we have; they would be able to construct straight lines, even though they were constrained to move in curves only; they would be able to define the nature of a space of three dimensions and would probably locate in the third dimension their gods and the abode of spirits. I insist that not sense experience, but *a priori* considerations, teach us the notions of straight lines.

The truth is that we tridimensional beings actually do live on a sphere, and we cannot get away from it. What is the highest flight of an æronaut and the deepest descent into a mine if measured by the radius of the earth? If we made an exact imitation of our planet, a yard in diameter, it would be like a polished ball, and the highest elevations would be less than a grain of sand; they would not be noticeable were it not for a difference in color and material.

When we become conscious of the nature of our habitation, we do not construct *a priori* conceptions accordingly, but feel limited to a narrow surface and behold with wonder the infinitude of space beyond. We can very well construct other *a priori* notions which would be adapted to one, or two, or four-dimensional worlds, or to spaces of positive or of negative curvatures, for all these constructions are ideal; they are mind-made and we select from them the one that would best serve our purpose of space-measurement.

The claim is made that if we were four-dimensional beings, our present three-dimensional world would appear to us as flat and shallow, as the plane is to us in our present tridimensional predicament. That statement is true, because it is conditioned by an "if." And what pretty romances have been built upon it! I remind my reader only of the ingenious story *Flatland, Written by a*

Square, and portions of Wilhelm Busch's charming tale *Edward's Dream*[1]; but the worth of conditional truths depends upon the assumption upon which they are made contingent, and the argument is easy enough that if things were different, they would not be what they are. If I had wings, I could fly; if I had gills I could live under water; if I were a magician I could work miracles.

INFINITUDE.

The notion is rife at present that infinitude is self-contradictory and impossible. But that notion originates from the error that space is a thing, an objective and concrete reality, if not actually material, yet consisting of some substance or essence. It is true that infinite things cannot exist, for things are always concrete and limited; but space is pure potentiality of concrete existence. Pure space is materially considered nothing. That this pure space (this apparent nothing) possesses some very definite positive qualities is a truth which at first sight may seem strange, but on closer inspection is quite natural and will be conceded by every one who comprehends the paramount significance of the doctrine of pure form.

Space being pure form of extension, it must be infinite, and infinite means that however far we go, in whatever direction we choose, we can go farther, and will never reach an end. Time is just as infinite as space. Our sun will set and the present day will pass away, but time will not stop. We can go backward to the beginning, and we must ask what was before the beginning. Yet suppose we could fill the blank with some hypothesis or another, mythological or metaphysical, we would not come to an absolute beginning. The same is true as to the end. And if the universe broke to pieces, time would continue, for even the duration in which the world would lie in ruins would be measurable.

Not only is space as a totality infinite, but in every part of space we have infinite directions.

What does it mean that space has infinite directions? If you lay down a direction by drawing a line from a given point, and continue to lay down other directions, there is no way of exhausting your possibilities. Light travels in all directions at once; but "all directions" means that the whole extent of the surroundings of a source of light is agitated, and if we attempt to gather in the whole by picking up every single direction of it, we stand before a task that cannot be finished.

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[1] *The Open Court*, Vol. VIII., p. 4266 et seq.

In the same way any line, though it be of definite length, can suffer infinite division, and the fraction $\frac{1}{3}$ is quite definite while the same amount if expressed in decimals as 0.333...., can never be completed. Light actually travels in all directions, which is a definite and concrete process, but if we try to lay them down one by one we find that we can as little exhaust their number as we can come to an end in divisibility or as we can reach the boundary of space, or as we can come to an ultimate number in counting. In other words reality is actual and definite but our mode of measuring it or reducing it to formulas admits of a more or less approximate treatment only, being the function of an infinite progress in some direction or other. There is an objective *raison d'être* for the conception of the infinite, but our formulation of it is subjective, and the puzzling feature of it originates from treating the subjective feature as an objective fact.

These considerations indicate that infinitude does not appertain to the thing, but to our method of viewing the thing. Things are always concrete and definite, but the relational of things admits of a progressive treatment. Space is not a thing, but the relational feature of things. If we say that space is infinite, we mean that a point may move incessantly and will never reach the end where its progress would be stopped.

There is a phrase current that the finite cannot comprehend the infinite. Man is supposed to be finite, and the infinite is identified with God or the Unknowable, or anything that surpasses the comprehension of the average intellect. The saying is based upon the prejudicial conception of the infinite as a realised actuality, while the infinite is not a concrete thing, but a series, a process, an aspect, or the plan of action that is carried on without stopping and shall not, as a matter of principle, be cut short. Accordingly, the infinite (though in its completeness unactualisable) is neither mysterious nor incomprehensible, and though mathematicians be finite, they may very successfully employ the infinite in their calculations.

I do not say that the idea of infinitude presents no difficulties, but I do deny that it is a self-contradictory notion and that if space must be conceived to be infinite, mathematics will sink into mysticism.

GEOMETRY REMAINS A PRIORI.

Those of our readers who have closely followed our arguments will now understand how in one important point we cannot accept Mr. B. A. W. Russell's statement as to the main result of the metageometrical inquisitions. He says:

> "There is thus a complete divorce between geometry and the study of actual space. Geometry does not give us certain knowledge as to what exists. That peculiar position which geometry formerly appeared to occupy, as an *a priori* science giving knowledge of something actual, now appears to have been erroneous. It points out a whole series of possibilities, each of which contains a whole system of connected propositions; but it throws no more light upon the nature of our space than arithmetic throws upon the population of Great Britain. Thus the plan of attack suggested by non-Euclidean geometry enables us to capture the last stronghold of those who attempt, from logical or *a priori* considerations, to deduce the nature of what exists. The conclusion *suggested* is, that no existential proposition can be deduced from one which is not existential. But to *prove* such a conclusion would demand a treatise upon all branches of philosophy."[1]

It is a matter of course that the single facts as to the population of Great Britain must be supplied by counting, and in the same way the measurement of angles and actual distances must be taken by *a posteriori* transactions; but having ascertained some lines and angles, we can (assuming our data to be correct) calculate other items with absolute exactness by purely *a priori* argument. There is no need (as Mr. Russell puts it) "from logical or *a priori* considerations to deduce the nature of what exists,"—

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[1] In *the new volumes of the Encyclopædia Britannica*, Vol. XXVIII., of the complete work, *s. v.* Geometry, Non-Euclidean, p. 674.

which seems to mean, to determine special features of concrete instances. No one ever assumed that the nature of particular cases, the qualities of material things, or sense-affecting properties, could be determined by *a priori* considerations. The real question is, whether or not the theorems of space relations and, generally, purely formal conceptions, such as are developed *a priori* in geometry and kindred formal sciences, will hold good in actual experience. In other words, can we assume that form is an objective quality, which would imply that the constitution of the actual world must be the same as the constitution of our purely *a priori* sciences? We answer this latter question in the affirmative.

We cannot determine by *a priori* reasoning the population of Great Britain. But we can *a posteriori* count the inhabitants of several towns and districts, and determine the total by addition. The rules of addition, of division, and multiplication can be relied upon for the calculation of objective facts.

Or to take a geometrical example. When we measure the distance between two observatories and also the angles at which at either end of the line thus laid down the moon appears in a given moment, we can calculate the moon's distance from the earth; and this is possible only on the assumption that the formal relations of objective space are the same as those of mathematical space. In other words, that our *a priori* mathematical calculations can be made to throw light upon the nature of space,—the real objective space of the world in which we live.

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The result of our inquisition is quite conservative. It reëstablishes the apriority of mathematical space, yet in doing so it justifies the method of metaphysicians in their constructions of the several non-Euclidean systems. *A priori* considered, the latter have equal rights with plane geometry, but for all that Euclidean geometry, which in the parallel theorem takes the bull by the horns, will remain classical forever, for after all the non-Euclidean systems cannot avoid developing the notion of the straight line or other even boundaries. Any geometry could, within its own premises, be utilised for a determination of objective space; but we will naturally give the preference to plane geometry, not because it is truer, but because it is simpler and will therefore be more serviceable.

How an ideal (and apparently purely subjective) construction can give us any information of the objective constitution of things, viz., so far as space-relations are concerned seems mysterious but the problem is solved if we bear in mind the objective nature of the *a priori*,—a subject which we have discussed in a previous chapter.[1]

SENSE-EXPERIENCE AND SPACE.

We have learned that sense-experience cannot be used as a source from which we construct our fundamental notions of geometry, yet sense-experience justifies them.

Experience can verify *a priori* constructions (as, e. g., tridimensionality is verified in Newton's laws), and can never change them. If experience does not tally with our calculations, we have either made a mistake or made a wrong observation. For our *a priori* conceptions hold good for any conditions, and their theory can be as little wrong as reality can be inconsistent.

However, some of the most ingenious thinkers and great mathematicians do not conceive of space as mere potentiality of existence, which renders it formal and purely *a priori*, but think of it as a concrete reality, as though it were a big box, presumably round, like an immeasurable sphere. If it were such, space would be (as Riemann says) boundless but not infinite, for we cannot find a boundary on the surface of a sphere, and yet the sphere has a finite surface that can be expressed in definite numbers.

I should like to know what Riemann would call that something which lies outside of his spherical space. Would the name "province of the extraspatial" perhaps be an appropriate term? I do not know how we can rid ourselves of this enormous portion of unutilised outside room. Strange though it may seem, this space-conception of Riemann counts among its advocates mathematicians of first rank, among whom I will here mention only the name of Sir Robert Ball.

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[1] See also the author's exposition of the problem of the *a priori* in his edition of *Kant's Prolegomena*, pp. 167–240.

It will be interesting to hear a modern thinker who is strongly affected by metageometrical studies, on the nature of space. Mr. Charles S. Peirce, an uncommonly keen logician and an original thinker of no mean repute, proposes the following three alternatives. He says:

> "First, space is, as Euclid teaches, both *unlimited* and *immeasurable*, so that the infinitely distant parts of any plane seen in perspective appear as a straight line, in which case the sum of the three angles of a triangle amounts to 180°; or,
>
> "Second, space is *immeasurable* but *limited*, so that the infinitely distant parts of any plane seen in perspective appear as a circle, beyond which all is blackness, and in this case the sum of the three angles of a triangle is less than 180° by an amount proportional to the area of the triangle; or
>
> "Third, space is *unlimited* but *finite* (like the surface of a sphere), so that it has no infinitely distant parts; but a finite journey along any straight line would bring one back to his original position, and looking off with an unobstructed view one would see the back of his own head enormously magnified, in which case the sum of the three angles of a triangle exceeds 180° by an amount proportional to the area.
>
> "Which of these three hypotheses is true we know not. The largest triangles we can measure are such as have the earth's orbit for base, and the distance of a fixed star for altitude. The angular magnitude resulting from subtracting the sum of the two angles at the base of such a triangle from 180° is called the star's *parallax*. The parallaxes of only about forty stars have been measured as yet. Two of them come out negative, that of Arided (α Cycni), a star of magnitude 1½, which is —0."082, according to C. A. F. Peters, and that of a star of magnitude 7¾, known as Piazzi III 422, which is —0."045 according to R. S. Ball. But these negative parallaxes are undoubtedly to be attributed to errors of observation; for the probable error of such a determination is about ± 0."075, and it would be strange indeed if we were to be able to see, as it were, more than half way round space, without being able to see stars with larger negative parallaxes. Indeed, the very fact that of all the parallaxes measured only two come out negative would be a strong argument that the smallest parallaxes really amount to +0."1, were it not for the reflexion that the publication of other negative parallaxes may have been suppressed. I think we may feel confident that the parallax of the furthest star lies somewhere between —0."05 and +0."15, and within another century our grandchildren will surely know whether the three angles of a triangle are greater or less than 180°,—that they are *exactly* that amount is what nobody ever can be justified in concluding. It is true that according to the axioms of geometry the sum of the

three sides of a triangle are precisely 180°; but these axioms are now exploded, and geometers confess that they, as geometers, know not the slightest reason for supposing them to be precisely true. They are expressions of our inborn conception of space, and as such are entitled to credit, so far as their truth could have influenced the formation of the mind. But that affords not the slightest reason for supposing them exact." (*The Monist*, Vol. I., pp. 173–174.)

Now, let us for argument's sake assume that the measurements of star-parallaxes unequivocally yield results which indicate that the sum of the angles in cosmic triangles is either a trifle more or a trifle less than 180°; would we have to conclude that cosmic space is curved, or would we not have to look for some concrete and special cause for the aberration of the light? If the moon is eclipsed while the sun still appears on the horizon, it proves only that the refraction of the solar rays makes the sun appear higher than it really stands, if its position is determined by a straight line, but it does not refute the straight line conception. Measurements of star parallaxes (if they could no longer be accounted for by the personal equation of erroneous observation), may prove that ether can slightly deflect the rays of light, but it will never prove that the straight line of plane geometry is really a curve. We might as well say that the norms of logic are refuted when we make faulty observations or whenever we are confronted by contradictory statements. No one feels called upon, on account of the many lies that are told, to propose a theory on the probable curvature of logic. Yet seriously speaking, in the province of pure being the theory of a curved logic has the same right to a respectful hearing as the curvature of space in the province of the scope of pure motility.

Ideal constructions, like the systems of geometry, logic, etc., cannot be refuted by facts. Our observation of facts may call attention to the logical mistakes we have made, but experience cannot overthrow logic itself or the principles of thinking. They bear their standard of correctness in themselves which is based upon the same principle of consistency that pervades any system of actual or purely ideal operations.

But if space is not round, are we not driven to the other horn of the dilemma that space is infinite?

Perhaps we are. What of it? I see nothing amiss in the idea of infinite space.

By the by, if objective space really is curved, would not the probability be one to infinity that its twist can scarcely be limited to the curvature of a circle, but will deviate as well in some other direction so that the straightest line would be spiral? And the spiral is as infinite as the straight line. Obviously, curved space does not get rid of infinitude; besides the infinitely small, which would not be thereby eliminated, is not less troublesome than the infinitely great.

THE TEACHING OF MATHEMATICS.

Euclid avoided the word axiom and I believe with Grassmann, that its omission in the *Elements* is not accidental but well-considered intention. The introduction of the term among Euclid's successors is due to a lack of clearness as to the nature of geometry and the conditions through which its fundamental notions originate.

It may be a flaw in the Euclidean *Elements* that the construction of the plane is presupposed, but it does not invalidate the details of his glorious work which will forever remain classical.

The invention of other geometries can only serve to illustrate the truth that all geometries, the plane geometry of Euclid included, are *a priori* constructions, and were not for obvious reasons Euclid's plane geometry preferable, other systems might as well be employed for the purpose of space-determination. Neither homaloidality nor curvature belongs to space; they belong to the several systems of manifolds that can be invented for the determination of the juxtaposition of things, called space.

If I had to rearrange the preliminary expositions of Euclid, I would state first the *Common Notions* which embody those general principles of Pure Reason and are indispensable for geometry. Then I would propose the *Postulates* which set forth our own activity (viz., the faculty of construction) and the conditions under which we intend to carry out our operations, viz., the obliteration of all particularity, characterisable as "anyness of motion," and finally I would lay down the *Definitions* as the most elementary constructions which are to serve as tools and objects for experiment in the further expositions of geometry. There would be no mention of axioms, nor would we have to regard anything as an assumption or an hypothesis.

Professor Hilbert has methodically arranged the principles that underlie mathematics, and the excellency of his work is universally recognised.[1] It is a pity, however, that he retains the term "axiom," and we would suggest replacing it by some other appropriate word. "Axiom" with Hilbert does not mean an obvious truth that does not stand in need of proof, but principle, or rule, viz., a formula describing certain general characteristic conditions.

Mathematical space is an ideal construction, and as such it is *a priori*. But its apriority is not as rigid as is the apriority of logic. It presupposes not only the rules of pure reason but also our own activity (viz., pure motility) both being sufficient to create any and all geometrical figures *a priori*.

Boundaries that are congruent with themselves being limits that are unique recommend themselves as standards of measurement. Hence the significance of the straight line, the plane, and the right angle.

The theorem of parallels is only a side issue of the implications of the straight line.

The postulates that figures of the same relations are congruent in whatever place they may be, and also that similar figures can be drawn to any figure, is due to our abstraction which creates the condition of anyness.

The teaching of mathematics, now utterly neglected in the public schools and not specially favored in the high schools, should begin early, but Euclid's method with his pedantic propositions and proofs should be replaced by construction work. Let children begin geometry by doing, not by reasoning. The reasoning faculties are not yet sufficiently developed in a child. Abstract reasoning is tedious, but if it comes in as an incidental aid to construction, it will be welcome. Action is the main-spring of life and the child will be interested so long as there is something to achieve.[1]

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[1] *The Foundations of Geometry*, The Open Court Pub. Co., Chicago, 1902.

Lines must be divided, perpendiculars dropped, parallel lines drawn, angles measured and transferred, triangles constructed, unknown quantities determined with the help of proportion, the nature of the triangle studied and its internal relations laid down and finally the right-angled triangle computed by the rules of trigonometry, etc. All instruction should consist in giving *tasks to be performed, not theorems to be proved;* and the pupil should find out the theorems merely because he needs them for his construction.

In the triangle as well as in the circle we should accustom ourselves to using the same names for the same parts.[2]

Every triangle is ABC. The angle at A is always α, on B β, on C γ. The side opposite A is a, opposite B b, opposite C c. The three heights are h_a, h_b, h_c. The lines that from A, B, and C pass through the center of gravity to the middle of the opposite sides I propose to call gravitals and would designate them g_a, g_b, g_c. The perpendiculars erected upon the middle of the sides meeting in the center of the circumscribed circle are p_a, p_b, p_c. The lines that divide the angles α, β, γ and meet in the center of the inscribed circle I propose to call "dichotoms" (from διχότομος[3]) and would designate them as d_a, d_b, d_c. The radius of the circumscribed circle is r, of the inscribed circle ρ, and the radii of the three ascribed circles are ρ_a, ρ_b, ρ_c. The point where the three heights meet is H, where the three gravitals meet R, where the three dichotoms meet O.[4] The stability of designation will be indispensable for a clear comprehension of these important interrelated parts.

EPILOGUE.

While matter is eternal and energy is indestructible, forms change; yet there is a feature in the changing forms of matter and energy that does not change. It is the norm that determines the nature of all formations, commonly called law or uniformity.

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[1] Cp. the author's article "Anticipate the School" (*Open Court*, 1899, p. 747).

[2] Such was the method of my teacher Prof. Hermann Grassmann.

[3] I purposely avoid the term *median* which is sometimes used in the sense of gravitals, sometimes as dichotoms. Thus the word median remains reserved for a general use.

[4] The capital of the Greek ρ is objectionable, because it cannot be distinguished from the Roman P.

The law of form is the condition that dominates our thinking and constitutes the norm of all formation, the principle of consistency which underlies our ideas of sameness, uniformity, rule, etc. This norm is not a concrete fact of existence but the universal feature that permeates both the anyness of our mathematical constructions and the anyness of objective conditions. Its application produces in the realm of mind the *a priori*, and in the domain of facts the uniformities of events which our scientists reduce to formulas, called laws of nature. On a superficial inspection it is pure nothingness, but in fact it is universality, eternality, and omnipresence; and it is the factor objectively of the world order and subjectively of science, the latter being man's capability of reducing the innumerable sense-impressions of experience to a methodical system of knowledge.

Faust, seeking the ideal of beauty, is advised to search for it in the domain of the eternal types of existence, which is the omnipresent Nowhere, the ever enduring Never. Mephistopheles calls it the Naught.[1] The norm of being, the foundation of natural law, the principle of thinking, is non-existent to Mephistopheles, and in his way Mephistopheles is right: but it is at the same time the rock of ages, it is the divinity of existence, and we might as well answer with a slight change of words, replacing "All" by "God," thus intensifying the meaning of Faust's reply:

> "'Tis in thy naught I hope to find my God."

The norm of Pure Reason, the factor that shapes the world, the eternal Logos, is omnipresent and eternal. It is God. The laws of nature have not been fashioned by a creator, they are part and parcel of the creator himself. Plutarch tells us that Plato said that God was always geometricising.[2] In other words, the purely formal theorems of mathematics and logic are the thoughts of God. Our thoughts are fleeting, but God's thoughts are eternal and omnipresent verities. They are intrinsically necessary, universal, immutable, and the standard of truth and right.

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[1] Cf. above the chapter on "The Apriori and the Purely Formal."

[2] Plutarchus *Convivia*, VIII., 2: *πῶς Πλάτων ἔλεγε τὸν Θεὸν ἀεὶ γεωμετρεῖν*. Having hunted in vain for the famous passage, I am indebted for the reference to Professor Ziwet of Ann Arbor, Mich.

We repeat, matter is eternal and energy is indestructible, but there is nothing divine in either matter or energy. That which constitutes the divinity of the world is the eternal principle of the laws of existence. That is the creator of the cosmos, the norm of truth, and the standard of right and wrong. If incarnated in living beings, it produces mind, and it continues to be the source of inspiration for aspiring mankind, a refuge of the struggling and storm-tossed sailors on the ocean of life, and the holy of holies of the religious devotee and worshipper.

The norms of logic and of mathematics are uncreate and uncreatable, they are irrefragable and immutable, and no power on earth or in heaven can change them. We can imagine that the world was made by a great world builder, but we cannot think that logic or arithmetic or geometry was ever fashioned by either man, or ghost, or god. Here is the rock on which the old-fashioned theology and all mythological God-conceptions must founder. If God were a being like man, if he had created the world as an artificer makes a tool, or a potter shapes a vessel, we would have to confess that he is a limited being. He would be infinitely greater and more powerful than man, but he would, as much as man, be subject to the same eternal laws, and he would, as much as human inventors and manufacturers, have to mind the multiplication tables, the theorems of mathematics, and the rules of logic.

Happily this conception of the deity may fairly well be regarded as antiquated. We know now that God is not a big individual, like his creatures, but that he is God, creator, law, and ultimate norm of everything. He is not personal but superpersonal. The qualities that characterise God are omnipresence, eternality, intrinsic necessity, etc., and surely wherever we find them we should take off our shoes, for whenever we face eternal verities it is a sign that we are in the presence of God,—not of a mythological God, but the God of the cosmic order, the God of mathematics and science, the God of ethics. So long as we can trace law in nature, as there is a norm of truth and untruth, and a standard of right and wrong, we need not turn atheists, even though the traditional conception of God is not yet free from crudities and mythological adornments. It will be by far preferable to purify our conception of God and replace the traditional notion which during the unscientific age of human development served man as a useful surrogate, by a new conception of God, that should be higher, and nobler, and better, because truer.

EDITOR.